Enrollment No: ______________________ Exam Seat No:______________________

C. U. SHAH UNIVERSITY

Winter Examination-2021

Subject Name: Metric Space

Subject Code: 4SC05MES1 **Branch: B.Sc. (Mathematics)**

Semester: 5 **Date: 20/12/2021** **Time: 11:00 To 02:00** **Marks: 70**

Instructions:

(1) Use of Programmable calculator & any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

Q-1 **Attempt the following questions:** **[14]**

a) Let (X, d) be a discrete metric space and $r > 1$,then $S_r(x) =$ _____ **(01)**
1) $\{x\}$
2) X
3) $(0,1)$
4) $[0,1]$

b) Which of the following subset of $\boldsymbol{R}$is neither open nor closed? **(01)**
1) $(0,1]$
2) R
3) $\emptyset$
4) None of these

c) If $E = (1,3)$ is subset of metric space $\boldsymbol{R}$then $E' =$ _____ **(01)**
1) $(1,3]$
2) $[1,3)$
3) $(1,3)$
4) $[1,3]$

d) Define :Connected sets **(01)**
e) Define : Limit Point **(01)**
f) Check whether the statement is true or false: If $A \subseteq B$ then $A^\circ \subseteq B^\circ$ **(01)**
g) Define : Metric Space **(01)**
h) Define Continuity in Metric space **(01)**
i) Find A° for $A = (0,4]$ **(01)**
j) Check whether the statement is true or false: Every closed and bounded subset of the real line is compact. **(01)**
k) Let $X = \boldsymbol{R}$and $A = \emptyset$ then find $int\ A$ and $ext\ A$. **(02)**
l) Show that the open interval $(0,1)$ with usual metric is not compact. **(02)**

Attempt any four questions from Q-2 to Q-8

Q-2		**Attempt all questions**	**[14]**
	a)	Let(X,d)be a metric space and $d_1: X \times X \to \boldsymbol{R}$ defined by $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$then prove that $d_1$is also a metric on X.	**(06)**
	b)	Let (X,d) be a metric space and $E \subset X$.If $'a'$ is a limit point of E then show that there are infinitely many points of E in every neighborhood of $'a'$.	**(04)**
	c)	Define: Closed Set .Show that every finite subset of metric space is closed.	**(04)**
Q-3		**Attempt all questions**	**[14]**
	a)	Prove: i) Arbitrary union of open sets of metric space is an open set. ii) Finite union of finite number of closed sets of metric space is a closed set.	**(06)**
	b)	Show that $(C[0,1],d)$ is metric space where $d(f,g) = \int_0^1 \lvert f(x) - g(x) \rvert \, dx$.	**(05)**
	c)	If (X,d) is a metric space and $A, B \subset X$ then show that$(A \cup B)' \subseteq A' \cup B'$.	**(03)**
Q-4		**Attempt all questions**	**[14]**
	a)	Let A, B be two subsets of metric space (X,d) then prove the following: i) $A = \bar{A}$ if and only if A is closed. ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$	**(06)**
	b)	Let $E_n = (c - \frac{1}{n}, c + \frac{1}{n})$ where $c \in \boldsymbol{N}$is constant and $n \in \boldsymbol{N}$.Compute $\bigcup_{n=1}^{\infty} E_n$and $\bigcap_{n=1}^{\infty} E_n$and determine whether they are open or closed?	**(05)**
	c)	Check whether the function $d(x,y) = \lvert x^2 - y^2 \rvert$for $\forall\, x, y \in \boldsymbol{R}$ is metric space over $\boldsymbol{R}$ or not.	**(03)**
Q-5		**Attempt all questions**	**[14]**
	a)	Let (X,d) be a metric space. If $\{x_n\}$ is convergent sequence of points of X then show that $\{x_n\}$ is cauchy sequence in X. What can you say about its converse?Justify.	**(06)**
	b)	Let (X,d_1) and (Y,d_2) be any two metric space,then prove that $f: X \to Y$ is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for every $A \subset X$.	**(05)**
	c)	Let $f: [a,b] \to [a,b]$ be differentiable, assume that $\lvert f'(x) \rvert \leq k$ where $x \in X$ and $k < 1$.Show that f is contraction mapping.	**(03)**
Q-6		**Attempt all questions**	**[14]**
	a)	Prove that any contraction mapping f of non-empty complete metric space (X,d) into itself has a unique fixed point.	**(06)**
	b)	Let (X,d_1) and (Y,d_2) be any two metric space,then prove that $f: X \to Y$ is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.	**(06)**
	c)	Give an example of subsets A and B of metric space $\boldsymbol{R}$such that $(A \cap B)' \neq A' \cap B'$.	**(02)**
Q-7		**Attempt all questions**	**[14]**
	a)	Let (X,d) be a complete metric space and $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$,then show that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.	**(07)**

b) Show that every compact subset A of metric space (X, d) is bounded. **(04)**

c) Which of the following sets are open sets?Justify. **(03)**

i) $A = \{1,2,3,8\}$ on $\boldsymbol{R}$

ii) $A = \{(x, y) \in \boldsymbol{R}^2 : y > x\}$ on $\boldsymbol{R}^2$

iii) $A = (0,1)$ on $\boldsymbol{R}$

Q-8 **Attempt all questions** **[14]**

a) For a non-empty subset A of metric space (X, d) show that the function $f: X \rightarrow \boldsymbol{R}$defined by $f(x) = d(x, A)$, $x \in X$ is uniformly continuous. Also show that $f(x) = 0$ if and only if $x \in \bar{A}$. **(07)**

b) Let A be connected subset of metric space X and B be a subset of X such that $A \subseteq B \subseteq \bar{A}$ then show that B is also connected. **(05)**

c) Show that the sets $A = (1,2)$ and $B = (2,3)$ are separated sets of metric space $\boldsymbol{R}$. **(02)**